Renormalization Approach to Uniform Asymptotics of Differential Equations

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These are notes written after teaching Physics 231a "Methods of Mathematical Physics" at UCLA. I was inspired to write them to better understand the fascinating renormalization group methods of Chen-Goldenfeld-Oono [1] for studying perturbation theory of ordinary differential equations. Renormalization is one of the pillars of physics, and I feel this particular application of the idea is very simple and distills some of the fundamental concepts which are crucial in quantum field theory. It is also a very effective method, as emphasized in [1], and has been used to solve important problems in singular perturbation theory of differential equations. My goal in this note is to simplify some of the presentation in [1, 2] and explain why the method works.

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I. TWO ILLUSTRATIVE EXAMPLES

A. A simple boundary layer problem

This example was described in [2].

The problem we want to solve for $y(u)$ is

$$y'' + y' = -\epsilon(y + y') \qquad y(0) = 0, y(1/\epsilon) = 1 \tag{1.1}$$

(prime denotes dy/du). We set up a regular perturbation theory calculation

$$y = y_0 + \epsilon y_1 + O(\epsilon^2). \tag{1.2}$$

We find

$$\begin{aligned} y_0 &= A_0 + B_0 e^{-u} \\ y_1 &= -A_0 u + A_1 + B_1 e^{-u}. \end{aligned} \tag{1.3}$$

At this stage, we have

$$y = A_0 + B_0 e^{-u} + \epsilon\left(-A_0 u + A_1 + B_1 e^{-u}\right) + O(\epsilon). \tag{1.4}$$

Notice that we can always absorb A_1, B_1 into A_0, B_0 since at each step we solve the same homogeneous equation, so we can simplify slightly to

$$y = (A_0 + \epsilon A_1) + (B_0 + \epsilon B_1)e^{-u} - \epsilon A_0 u + O(\epsilon). \tag{1.5}$$

But this expression as an approximation in ϵ is not uniformly valid in u because when $u \sim 1/\epsilon$, $-\epsilon A_0 u$ becomes comparable to the leading terms. It is known as a "secular term", and is somewhat analogous to a divergence in quantum field theory.

To perform renormalization, we introduce a new variable μ, functions $A(\mu)$, $B(\mu)$, and consider the expression

$$Y(u, \mu) = A(\mu) + B(\mu)e^{-u} - \epsilon A(\mu)(u - \mu) + O(\epsilon^2). \tag{1.6}$$

This expression satisfies the differential equation as a function of u (up to higher order terms) for arbitrary choices of $A(\mu)$, $B(\mu)$, since the added "counterterm"

$$\epsilon A(\mu)\mu \tag{1.7}$$

is a homogeneous solution to the equation for y_1 (it amounts to setting $A_1 = A(\mu)\mu$, which is a constant as far as u is concerned). We aim to choose $A(\mu)$, $B(\mu)$ such that the "renormalization equation" holds:

$$\frac{\partial}{\partial\mu} Y(u,\mu) = 0. \tag{1.8}$$

(at least up to next order). Then $Y(u,\mu)$ will be independent of μ to this order and we can consider

$$Y(u,\mu) = Y(u,u) = A(u) + B(u)e^{-u} + O(\epsilon^2). \tag{1.9}$$

This remains a solution to the differential equation (to this order) thanks to the renormalization equation (1.8), and the secular term has now canceled the counterterm! This is then a *uniformly valid* approximation for $y(u)$ as $\epsilon \to 0$.

A key assumption which makes the renormalization equation solvable order by order is[1]

$$\frac{dA}{d\mu} = \frac{dB}{d\mu} = O(\epsilon) \tag{1.10}$$

Under this assumption, we have

$$\frac{\partial}{\partial\mu} Y = \frac{dA}{d\mu} + \frac{dB}{d\mu} e^{-u} + \epsilon A + O(\epsilon^2) \tag{1.11}$$

This has to hold for all u, so we get separate equations for A and B:

$$\begin{aligned} \frac{dA}{d\mu} &= -\epsilon A + O(\epsilon^2) \\ \frac{dB}{d\mu} &= 0 + O(\epsilon^2). \end{aligned} \tag{1.12}$$

This yields

$$\begin{aligned} A &= \bar{A}e^{-\epsilon u} + O(\epsilon^2) \\ B &= \bar{B} + O(\epsilon^2), \end{aligned} \tag{1.13}$$

[1] This relation is also dictated to us by dominant balance.

where $\bar{A}$ and $\bar{B}$ are constants, to be set by the boundary conditions. Our uniformly valid approximation is then

$$Y(u,u) = \bar{A}e^{-u\epsilon} + \bar{B}e^{-u} + O(\epsilon^2). \tag{1.14}$$

For this simple problem, this turns out to be the exact solution.

B. Rayleigh equation

This example was discussed in [1].

The "Rayleigh equation" for $y(t)$ is

$$y'' + y = \epsilon(y' - \frac{1}{3}(y')^3). \tag{1.15}$$

This is known to have a circular limit cycle of amplitude 2. We set up a regular perturbation series

$$\begin{gathered} y_0'' + y_0 = 0 \\ y_0 = A_0 \sin(t+\theta_0) \\ y_1'' + y_1 = \epsilon(A_0 \sin(t+\theta_0) - A_0^3 \sin^3(t+\theta_0)) \\ y_1 = \frac{R_0}{2}(1 - \frac{R_0^2}{4})t\sin(t+\theta_0) + \frac{R_0}{96}\cos(3(t+\theta_0)). \end{gathered} \tag{1.16}$$

We have only kept the particular solution for y_1, since the homogeneous solution can be absorbed into A_0 and θ_0, as in the previous example. We will treat the boundary conditions after applying the renormalization method to deal with the secular term $\frac{R_0}{2}(1 - \frac{R_0^2}{4})t\sin(t+\theta_0)$.

We introduce the variable τ and the renormalized approximation

$$\begin{gathered} Y(t,\tau) = R(\tau)\sin(t+\theta(\tau)) \\ +\epsilon(\frac{R(\tau)}{2}(1 - \frac{R(\tau)^2}{4})(t-\tau)\sin(t+\theta(\tau)) + \frac{R(\tau)}{96}\cos(3(t+\theta(\tau)))) + O(\epsilon^2) \end{gathered} \tag{1.17}$$

Note that the new term

$$-\epsilon\frac{R(\tau)}{2}(1 - \frac{R(\tau)^2}{4})\tau\sin(t+\theta(\tau)) \tag{1.18}$$

appears as a homogeneous solution for y_1, so we are free to include it, and $Y(t,\tau)$ satisfies the differential equation as a function of t up to $O(\epsilon^2)$.

We will derive the renormalization equation $\partial_\tau Y = 0$ under the assumptions

$$\frac{dR}{d\tau} = \frac{d\theta}{d\tau} = O(\epsilon) \tag{1.19}$$

as previously. Note that under this assumption, only the secular terms of order ϵ can contribute to the RG equation. We find

$$\frac{\partial}{\partial\tau}Y = \frac{dR}{d\tau}\sin(t+\theta) + R\frac{d\theta}{d\tau}\cos(t+\theta) - \epsilon\frac{R(\tau)}{2}(1-\frac{R(\tau)^2}{4})\sin(t+\theta(\tau)) + O(\epsilon^2). \tag{1.20}$$

The sine and cosine equations have to hold simultaneously, so we get

$$\begin{aligned} \frac{dR}{d\tau} &= \epsilon\frac{R(\tau)}{2}(1-\frac{R(\tau)^2}{4}) + O(\epsilon^2) \\ \frac{d\theta}{d\tau} &= O(\epsilon^2). \end{aligned} \tag{1.21}$$

The first equation can be solved by Mathematica, giving

$$R(\tau) = \frac{2}{\sqrt{1+Ce^{-\epsilon\tau}}} + O(\epsilon^2). \tag{1.22}$$

The renormalized uniform approximation is thus

$$y = \frac{2}{\sqrt{1+Ce^{-\epsilon t}}}\sin(t+\theta_0) - \epsilon\frac{1}{96}\frac{8}{(1+Ce^{-\epsilon t})^{3/2}}\cos 3(t+\theta_0) + O(\epsilon^2). \tag{1.23}$$

This reveals the circular limit cycle of radius 2, which can be seen from the first term as $t \to \infty$.

II. WHY DOES IT WORK?

The crucial step whose validity must be checked is that when we write the renormalized expression $Y(t,\tau)$, it remains a solution to the differential equation in t, at least to that order. Once that is the case, and the renormalization equation $\partial_\tau Y(t,\tau) = 0$ holds (again at least to the appropriate order), then we will be free to set $\tau = t$ to cancel the secular terms and obtain a uniformly valid approximation.

First of all, upgrading the constants of integration $A_0 \mapsto A(\tau)$ always results in a valid solution because τ may be regarded as constant.

Second, we have to make sure that the counterterm does not affect the differential equation to that order. We have seen in the examples above that it can be considered a particular homogeneous solution for y_1, so as long as this is true, $Y(t, \tau)$ will be a solution and renormalization will yield a uniformly valid asymptotic.

This will indeed be the case under mild assumptions. So long as the regular perturbation expansion can be applied to the differential equation, the equation for y_n will be linear of some order m, since y_n's always come with factors of ϵ^n. Let $h_1, \ldots, h_m$ be the homogeneous solutions to this linear equation. The equation for y_n is typically inhomogeneous, driven by solutions determined in previous orders of perturbation theory. However, again on mild conditions of those solutions (such as continuity), the solutions to y_n will be determined by "variation of parameters" and take the form

$$y_n(t) = \sum_{i=1}^{m} f_i(t) h_i(t) \tag{2.1}$$

for some functions $f_i(t)$. These functions are the source of secular terms when they are growing. We see that they may always be canceled by counterterms

$$-f_i(\tau) h_i(t), \tag{2.2}$$

which will be a solution to the homogeneous equation. Thus, we are free to add such counterterms and still have a solution to the differential equation. Note we are free to choose any convenient counterterm

$$-\sum_{i=1}^{m} g(\tau)_i h_i(t) \tag{2.3}$$

such that

$$f_i(t) - g_i(t) \tag{2.4}$$

is bounded by an $O(1)$ constant on the domain of interest.

III. MORE EXAMPLES

A. Unstable Spring Bifurcation

This is another example discussed in [1].

The equation is

$$y'' + y = \epsilon t y \qquad y(0) = 1, y'(0) = 0. \tag{3.1}$$

Regular perturbation theory produces

$$\begin{aligned} y_0 &= R_0 \cos(t + \theta_0) \\ y_1 &= \frac{R_0}{8}\left(2t\cos(t+\theta_0) + (2t^2 - 1)\sin(t+\theta_0)\right). \end{aligned} \tag{3.2}$$

This has two secular terms. We take

$$Y(t,\tau) = R(\tau)(1 + \frac{\epsilon}{4}(t-\tau))\cos(t+\theta) + \epsilon\frac{R(\tau)}{8}(2(t^2-\tau^2)-1)\sin(t+\theta) + O(\epsilon^2). \tag{3.3}$$

Both added terms are homogeneous solutions to the y_1 equation, so $Y(t,\tau)$ remains a solution to the differential equation in t to this order. The renormalization equations are

$$\begin{aligned} \frac{\partial}{\partial\tau}Y &= (\frac{dR}{d\tau} - \frac{\epsilon R}{4})\cos(t+\theta) + (-R\frac{d\theta}{d\tau} - \frac{\epsilon R\tau}{2})\sin(t+\theta) + O(\epsilon^2) \\ \frac{dR}{d\tau} &= \frac{\epsilon R}{4} + O(\epsilon^2) \\ \frac{d\theta}{d\tau} &= -\frac{\epsilon\tau}{2} + O(\epsilon^2). \end{aligned} \tag{3.4}$$

These have solutions

$$\begin{aligned} R(\tau) &= \bar{R}e^{\epsilon\tau/4} + O(\epsilon^2) \\ \theta(\tau) &= \bar{\theta} - \epsilon\tau^2/4 + O(\epsilon^2). \end{aligned} \tag{3.5}$$

Our renormalized approximation to this order is thus

$$y = \bar{R}e^{\epsilon t/4}\cos(t - \epsilon t^2/4 + \bar{\theta}) - \epsilon\frac{\bar{R}}{8}e^{\epsilon t/4}\sin(t - \epsilon t^2/4 + \bar{\theta}) + O(\epsilon^2). \tag{3.6}$$

Note that although the prefactor $e^{\epsilon t/4}$ is growing, this is still a uniformly valid approximation because comparing with the first term $\epsilon e^{\epsilon t/4} \ll e^{\epsilon t/4}$.

As an interesting aside, we can also apply WKB to this problem, and obtain a leading term

$$\frac{\bar{R}}{(1-\epsilon t)^{1/4}}\cos(t-\epsilon t^2/4+\bar{\theta}). \tag{3.7}$$

This approximation better matches the growth of amplitude as t approaches the critical value $t_c = 1/\epsilon$ where the system has a bifurcation. Note

$$(1-\epsilon t)^{-1/4} = e^{-\frac{1}{4}\log(1-\epsilon t)} = e^{\frac{1}{4}\epsilon t+O(\epsilon^2)}. \tag{3.8}$$

In [1], the authors suggest a change of variables called the Liouville-Green transformation to better capture this behavior.

[1] L.-Y. Chen, N. Goldenfeld, and Y. Oono, Physical Review E **54**, 376–394 (1996).

[2] T. Kunihiro, Progress of Theoretical Physics **94**, 503–514 (1995).